Université Paris Dauphine
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Introduction to Time series
TD5 - Linear projection

Exercice 1 Let X be a stationary process with mean $\mu \in \mathbb{R}$. We set $Y_t = X_t - \mu$. Show that, for all $t \in \mathbb{Z}$ and for all $p \in \mathbb{N}^*$,

$$\text{proj}(X_t, \text{Vect}\{1, X_{t-1}, \ldots, X_{t-p}\}) = \mu + \text{proj}(Y_t, \text{Vect}\{Y_{t-1}, \ldots, Y_{t-p}\}).$$

Exercice 2 Consider the following AR(2) equation:

$$X_t = Z_t + \frac{3}{4}X_{t-1} - \frac{1}{8}X_{t-2} \qquad \forall t \in \mathbb{Z}.$$

1. Show that there is a unique stationary solution X to this equation, which is causal and invertible.
2. Determine the progressive predictor of order p of X, for all $p \geq 1$.

Exercice 3 We consider the following ARMA(2,1) equation:

$$X_t = Z_t - \frac{1}{2}Z_{t-1} + X_{t-1} - \frac{1}{4}X_{t-2} \qquad \forall t \in \mathbb{Z}.$$

1. Show that this equation has a unique stationary solution X.
2. Show that X is the solution of an AR(1) equation and give its explicit formula.
3. Determine the progressive predictor of order p of X, for all $p \geq 1$.

Exercice 4 Let (X_t) be a centered stationary process and $p \in \mathbb{N}^*$. For all $t \in \mathbb{Z}$ and $h \in \mathbb{N}^*$ we set

$$\widehat{X}_{t,h} = \text{proj}(X_t, \text{Vect}\{X_{t-h}, \ldots, X_{t-h-p+1}\}).$$

1. Determine the equation that the coefficients $\theta_1, \ldots, \theta_p$ satisfy such that $\widehat{X}_{t,h} = \sum_{k=1}^{p} \theta_k X_{t-h-k+1}$.
2. Show that $\|X_t - \widehat{X}_{t,h}\|_2$ does not depend on t.
3. We assume that (X_t) is decorrelated at infinity (i.e., $\gamma_X(h) \to 0$ when $h \to +\infty$). Show that $(\widehat{X}_{t,h})$ tends to 0 in L^2 when $h \to +\infty$.

Exercice 5 (Deterministic process) A centered stationary process X is said to be deterministic if

$$X_t \in \overline{\text{Vect}\{X_s,\ s < t\}} \qquad \forall t \in \mathbb{Z}.$$

We recall that, for $n \in \mathbb{N}^*$, $\sigma_n^2 = \text{Var}(X_t - \text{proj}(X_t, H_{t-1,n})$ where $H_{t-1,n} = \text{Vect}(X_{t-k},\ k \in \{1, \ldots, n\}$. Let $\sigma_\infty := \lim_{n\ to\infty} \sigma_n$ (this limit exists because (σ_n) is non-increasing).

1. Show that X is deterministic if and only if $\sigma_\infty = 0$.

2. Give an example of a deterministic process and an example of a non-deterministic process.
3. Are AR(p) processes deterministic?

Exercice 6 We consider the general equation ARMA(p,q)

$$X_t = Z_t + \sum_{k=1}^{p} a_k X_{t-k} + \sum_{k=1}^{q} b_k Z_{t-k}$$

Assume that the polynomials $A(z) = 1 - \sum_{k=1}^{p} a_k z^k$ and $B(z) = 1 + \sum_{k=1}^{q} b_k z^k$ do not have a root inside the unit disk (all the roots are of modulus > 1).

1. Show that there exists $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ in ℓ^1 such that, for all $t \in \mathbb{Z}$,

$$X_t = \sum_{k=0}^{\infty} \beta_k Z_{t-k} \text{ et } Z_t = \sum_{k=0}^{\infty} \alpha_k X_{t-k}$$

2. For a given $n \in \mathbb{Z}$, we denote $H_n^X = \overline{\text{Vect}\{X_k,\ k \in \mathbb{Z} \text{ and } k \leq n\}}$. Show that

$$H_n^X = \overline{\text{Vect}\{Z_k,\ k \in \mathbb{Z} \text{ and } k \leq n\}}.$$

3. For $n \in \mathbb{N}^*$, set $\hat{X}_t = \text{proj}(X_t, H_n^X)$. Show that, for all $t \geq 1$, we have

$$\hat{X}_{t+n} = -\sum_{j=1}^{\infty} \alpha_j \hat{X}_{t+n-j} = \sum_{j=t}^{\infty} \beta_j Z_{t+n-j}$$

and

$$\text{var}(X_{t+n} - \hat{X}_{t+n}) = \sum_{j=0}^{t-1} \beta_j^2.$$

4. Compute $\hat{X}_{n+1}$ in the particular case AR(1) and MA(q).